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Weakly Connected k-fair Domination in Graphs

Doli-Jane U. Tejada, Rendon A. dela Cruz, Mae P. Militante, Berlita Y. Disca, Marchelle D. Matulac, Jeanifer B. Lanzuela

Mathematics Department, Mindanao State University, General Santos City, Philippines

Abstract: In a connected graph $G = (V(G), E(G))$, a dominating set $D \subseteq V(G)$ is a weakly connected k-fair dominating set ($kwfd$-set) in G if the weakly induced subgraph of D is connected and $|N_G(u) \cap D| = k$ for every $u \in V(G) \setminus D$ for some integer $k \geq 1$. The weakly connected k-fair domination number of G, denoted by $\gamma_{kwfd}(G)$ is the minimum cardinality of a weakly connected k- fair dominating set. In this paper, we study the weakly connected k- fair domination numbers of some families of graphs such as the complete graphs (K_n), paths (P_n), cycles (C_n), helm graphs (H_n) and complete bipartite graphs ($K_{m,n}$). We also characterize the weakly connected k- fair dominating sets in join $K_1 + H$, and the vertex corona $G \circ H$. Moreover, sufficient conditions for weakly connected k-fair dominating sets of edge corona $G \diamond H$ of graphs are obtained.

Keywords: complete bipartite, corona of graphs, cycle, helm, join of graphs, path, weakly connected k-fair domination

1 INTRODUCTION

In graph theory, one of the most extensively researched topics is the domination in graphs. For an in-depth understanding of domination concepts, valuable references include the works of Haynes et al. [9] and [10]. Let $G = (V(G), E(G))$ be an undirected connected graph. The set of neighbors of a vertex $u \in V(G)$ is called the open neighborhood of u and is denoted by $N_G(u)$ and the closed neighborhood of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The open neighborhood of $U \subseteq V(G)$ is $N_G(U) = \bigcup_{u\in U} N_G(u)$ and the closed neighborhood of U is $N_G[U] = N_G(U) \cup U$. A set $D \subseteq V(G)$ is a dominating set of G if for every $u \in V(G) \setminus D$, there exists $v \in D$ such that $uv \in E(G)$. The domination number of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G.

A subset D of $V(G)$ is called weakly connected if the subgraph $\langle D \rangle_w = (N_G[D], E_w)$ weakly induced by D is connected, where E_w consists of edges in G with at least one vertex in D. A dominating set $D \subseteq V(G)$ is a weakly connected dominating set of G if the subgraph, $\langle D \rangle_w$, weakly induced by D is connected. The weakly connected domination number of G, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set of G. The concept of weakly connected domination was introduced by Grossman [7] and was studied by Dunbar, et al. [2] where sharp upper and lower bounds for $\gamma_w(G)$ were obtained. Additional investigations on weakly connected domination surfaced in the literature [11,14,15,16].

A noteworthy variation, fair domination, emerged in 2012 through the contributions of Caro, Hansberg, and Henning [1]. Subsequent research delved into fair domination, as evidenced by studies in [3,4,5,6,8,17]. Maravilla et al. [12] and [13] extended this notion in 2014, introducing k-fair domination and characterizing k-fair dominating sets across various graph operations such as join, corona, composition, and cartesian product. Building upon these advancements, Usman et al. [18,19,20] explored connected k-fair domination and neighborhood connected k-fair domination in 2018 and 2019, respectively.

A dominating set $D \subseteq V(G)$ is a fair dominating set (fd-set) in G if for every two distinct vertices u and v from $V(G) \setminus D$, $|N_G(u) \cap D| = |N_G(v) \cap D|$. The fair domination number of G, denoted by $\gamma_{fd}(G)$, is the minimum cardinality of an fd-set. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called a minimum fair dominating set or a $\gamma_{fd}(G)$-set. A dominating set $D \subseteq V(G)$ is a k- fair dominating set if $|N_G(u) \cap D| = k$ for every $u \in V(G) \setminus D$ for some integer $k \geq 1$. The k-fair domination number of G, denoted $\gamma_{kfd}(G)$ is the minimum cardinality of a k -fair dominating set, abbreviated $k$fd-set. A k-fair dominating set of cardinality $\gamma_{kfd}(G)$ is called γ_{kfd} – set of G. The weakly connected k- fair domination is another variant of domination where it combines the concepts of domination and the idea of weakening the conditions of connectivity of vertices in a set.

A dominating set $D \subseteq V(G)$ is a weakly connected k -fair dominating set in G, abbreviated $k$wfd – set, if the subgraph $\langle D \rangle_w = (N_G[D], E_w)$ is connected and $|N_G(u) \cap D| = k$ for every $u \in V(G) \setminus D$ for some integer $k \geq 1$. The weakly connected k-fair domination number of G, denoted by $\gamma_{kwfd}(G)$ is the minimum cardinality of a weakly connected k- fair dominating set. The weakly connected fair domination number of G denoted by $\gamma_{wfd}(G)$ is defined as $\gamma_{wfd}(G) = \min\{\gamma_{kwfd}(G)\}$, where the minimum is taken over all integers k with $1 \leq k \leq n-1$. A weakly connected k- fair dominating set of cardinality $\gamma_{kwfd}(G)$ is called $\gamma_{kwfd}(G)$-set of G.

In this paper, we introduce and investigate the concept of weakly connected k- fair domination, where k is a positive integer. The formula for the weakly connected k-fair domination numbers of some families of graphs such as the complete graphs K_n, paths P_n, cycles C_n, helm graphs H_n and complete bipartite graphs $K_{m,n}$ are provided. The necessary and sufficient conditions for the weakly connected fair domination number of a graph is exactly equal to 2 is given. We characterize the weakly connected fair dominating sets in join $K_1 + H$, and the vertex corona $G \circ H$, where G is any connected graph and H is any graph. The sufficient conditions for the weakly connected k-fair dominating sets of the edge corona $G \diamond H$, where G is any connected graph and H is any graph are obtained.

The vertex set $V(G)$ of a connected graph G is itself an wfd- set. A weakly connected fair dominating set (wfd-set) in G is an $rwfd$-set for some integer $r \geq 1$.

The *disjoint union* of X and Y, denoted by $X \dot{\cup} Y$ (the symbol $\dot{\cup}$ denotes the disjoin union) is the set obtained by taking the union of X and Y treating each element in X as distinct from each element in Y. The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge set $E(G + H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv: u \in V(G), v \in V(H)\}$.

Remark 1.1 Let G and H be connected graphs. If $S \subseteq V(G + H)$ is nonempty, then $\langle S \rangle_w$ is connected.

2 MAIN RESULTS

Remark 2.1 Let G be any nontrivial connected graph. Then $1 \leq \gamma(G) \leq \gamma_{fd}(G) \leq \gamma_{wfd}(G) \leq \gamma_{kwfd}(G)$ for some integer $k \geq 1$.

Theorem 2.2 For any nontrivial connected graph of order n. Then $\gamma_{wfd}(G) \leq n - 1$.

Proof: The case $n = 2$ will mean that $G = K_2$ and hence $\gamma_{wfd}(G) \leq 1$. Let $n \geq 3$. Set $D = V(G) \setminus \{x\}$, where $D \cup \{x\} = V(G)$. Since G is connected, there exists $y \in D$ such that $xy \in E(G)$. Thus, D is a dominating set. Also, D is a fair dominating set in fact $|N_G(x) \cap D| = |N_G(x)|$. Let $u, v \in D$. Then $uv \in E(G)$ or $uv \notin E(G)$. If $uv \in E(G)$, then $\langle D \rangle_w$ is connected. If $uv \notin E(G)$. Then there exists $w \in V(G)$ such that $uw, wv \in E(G)$. Thus, there is a $u - v$ path for every $u, v \in D$. It follows that $\langle D \rangle_w$ is connected. Hence, D is a wfd-set in G. Therefore, by Remark 2.1, $\gamma_{wfd}(G) \leq |D| = n - 1$. ■

Theorem 2.3 Let G be a nontrivial connected regular graph of order $n > 2$. Then $\gamma_{wfd}(G) \leq n - 2$.

Proof: Let G be a connected graph. Further, let G be an m-regular graph. Since $n \geq 3$, we have $m \geq 2$. Let D be the minimum weakly connected dominating set of G. Then $\gamma_w(G) = |D| \leq n - 2$. Choose $u, v \in V(G) \setminus D$ and set $S = V(G) \setminus \{u, v\}$. Since $D \subseteq S$, S is a weakly dominating set of G. If $uv \in E(G)$, then $|N_G(u) \cap S| = m - 1 = |N_G(v) \cap S|$. On the other hand, if $uv \notin E(G)$, then $|N_G(u) \cap S| = m = |N_G(v) \cap S|$. It follows that S is a fair dominating set in G. Hence, S is a wfd-set in G. Consequently, $\gamma_{wfd}(G) \leq |S| = n - 2$. ■

Theorem 2.4. Let G be any connected graph. Then $\gamma_{wfd}(G) = 1$ if and only if $\gamma(G) = 1$.

Proof: If G is a trivial graph, then $\gamma(G) = \gamma_{wfd}(G) = |V(G)| = 1$. Let $|V(G)| \geq 2$. Suppose that $\gamma_{wfd}(G) = 1$. Then by Remark 2.1, $1 \leq \gamma(G) \leq \gamma_{wfd}(G) = 1$ implying that $\gamma(G) = 1$.

Conversely, suppose $\gamma(G) = 1$. Let $D = \{x\}$ be a minimum dominating set in G. Then for every $y \in V(G) \setminus D$, $N_G(y) \cap D = \{x\}$. It follows that for all $y, z \in V(G) \setminus D$ with $y \neq z$, we have $|N_G(y) \cap D| = |D| = 1 = |N_G(z) \cap D|$. Thus, D is a $1fd$-set of G. Also, the subgraph weakly induced by D is isomorphic to the join of K_1 and H, where H is any graph with $V(H) = V(G) \setminus \{x\}$. Thus, $\langle D \rangle_w$ is connected. It follows that D is a weakly connected fair dominating set in G. By Remark 2.1, $1 \leq \gamma_{wfd}(G) \leq |D| = 1$. Therefore, by Remark 2.1, $\gamma_{wfd}(G) = 1$. ■

The *path* P_n is a graph with distinct vertices $v_1, v_2, \ldots, v_n$ *and* edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. The *cycle* C_n is a graph with distinct vertices $a_1, a_2, \ldots, a_n$ and edges $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n, a_na_1$. The *helm* H_n is a graph formed from a wheel by attaching a pendant vertex at each of the vertices of the $n-$ cycle. Figure 1 shows the schematic diagrams of path P_n, where $n \geq 2$, cycle C_n, and helm H_n graphs, $n \geq 3$.

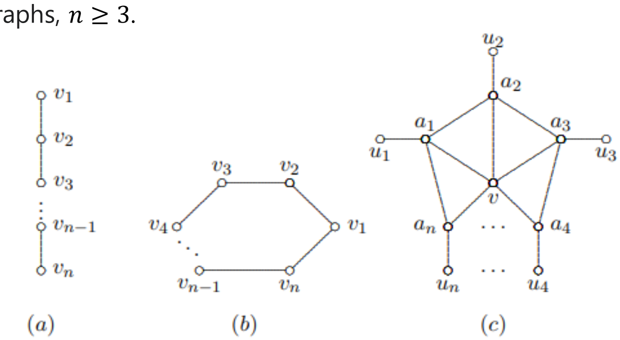

Figure 1: (a) A path P_n; (b) cycle C_n; and (c) helm H_n

Theorem 2.5. Let P_n be a path graph with $n \geq 2$. Then

$$\gamma_{wfd}(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & if \quad n < 5 \\ \left\lceil \frac{n+1}{2} \right\rceil, & if \quad n \geq 5 \end{cases}.$$

Proof: Let $P_n = [v_1, v_2, \ldots, v_n]$. Suppose first $n < 5$. For $n = 2$, $n = 3$, and $n = 4$ consider the following sets namely, $D_1 = \{v_1\}$, $D_2 = \{v_2\}$ and $D_3 = \{v_2, v_3\}$, respectively. Thus D_1, D_2 and D_3 are respectively the minimum wfd-set in P_n. Hence, $\gamma_{wfd}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Secondly, suppose $n \geq 5$. Consider $D_4 = \{v_1, v_3, \ldots, v_{n-2}, v_n\}$ when n is odd and

$D_5 = \{v_1, v_3, \ldots, v_{n-1}, v_n\}$ when n is even. Then $\langle D_4 \rangle_w = P_n$ and $\langle D_5 \rangle_w = P_n$. Thus, $\langle D_4 \rangle_w$ and $\langle D_5 \rangle_w$ are connected and 2-fair dominating set. Hence, D_4 is a wfd-set in P_n for odd integer n with $|D_4| = \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil$ and D_5 is a wfd-set in P_n for even integer n with $|D_5| = \frac{n}{2} + 1 = \left\lceil \frac{n+1}{2} \right\rceil$. Thus, $\gamma_{wfd}(P_n) \leq \left\lceil \frac{n+1}{2} \right\rceil$. Next, let D^* be a γ_{wfd}-set in P_n. Since D^* is fair dominating set in P_n, $v_1, v_n \in D^*$. Suppose $v_k \notin D^*$ for some $k \in \{2,3,\ldots,n-1\}$. Then $v_{k-1}, v_{k+1} \in D^*$ since denial of any of the two eventualities would produce an edge of P_n which is incident to none of the vertices in D^*, contradicting to the fact that $\langle D^* \rangle_w$ must be connected or must be a fair dominating set. In the case when n is odd, $D^* = \{v_1, v_3, \ldots, v_{n-2}, v_n\}$ where $|D^*| = \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil$. But if n is even, the property that each subset of $V(P_n)$ of the form $\{v_{k-1}, v_{k+1}\}$, $k = 2,3,\ldots,n-2$, always intersects with D^* implies that D^* contains an element from each of the subsets $\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{n-2}, v_{n-1}\}$. In this case, we have $|D^*| \geq 2 + \frac{n-2}{2} = \frac{n}{2} + 1 = \left\lceil \frac{n+1}{2} \right\rceil$. From both cases, we have, $\gamma_{wfd}(P_n) \geq \left\lceil \frac{n+1}{2} \right\rceil$. Therefore, $\gamma_{wfd}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$. ∎

Theorem 2.6. Let C_n be a cycle graph with $n \geq 3$. Then

$$\gamma_{wfd}(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof: Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Consider $D_1 = \{v_1, v_3, \ldots, v_{n-2}, v_n\}$ when n is even and $D_2 = \{v_1, v_3, \ldots, v_{n-1}, v_n\}$ when n is odd. Then $\langle D_1 \rangle_w = C_n$ and $\langle D_2 \rangle_w = C_n$. It follows that $\langle D_1 \rangle_w$ and $\langle D_2 \rangle_w$ are connected and 2-fair dominating set in C_n. Thus, D_1 and D_2 are wfd-set in C_n. Thus, $\gamma_{wfd}(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$. Next, let D^* be a γ_{wfd}-set in C_n. Since D^* is dominating set, $|D^*| \geq \left\lceil \frac{n}{2} \right\rceil$ for $n = 3,4$. On the other hand, for $n \geq 5$ any weakly connected fair dominating set in C_n is a $2fd$-set. It follows that $|D^*| \geq \left\lceil \frac{n}{2} \right\rceil$. Hence, $\gamma_{wfd}(C_n) = |D^*| \geq \left\lceil \frac{n}{2} \right\rceil$. Therefore, $\gamma_{wfd}(C_n) = \left\lceil \frac{n}{2} \right\rceil$. ∎

Corollary 2.7. Let H_n be a helm graph. Then

$$\gamma_{wfd}(H_n) = n + 1.$$

Proof: We denote the central vertex of a wheel by v and the vertices of n-cycle by a_i and the leaves of H_n by u_i, $1 \leq i \leq n$ (see Figure 1 (c)). Let $D = \{u_i : 1 \leq i \leq n\} \cup \{v\}$. Then $\langle D \rangle_w$ is connected as shown in Figure 2(a). Let $a, b \in V(H_n) \setminus D$. Then $|N_{H_n}(a) \cap D| = 2 = |N_{H_n}(b) \cap D|$. It follows that D is a fair dominating set in H_n. Thus, D is a wfd-set in H_n. Hence, $\gamma_{wfd}(H_n) \leq |D| = n + 1$. Next, let D' be a γ_{wfd}-set in H_n. Then $v \in D'$. Thus, we have either $D_1{}' = \{a_i : 1 \leq i \leq n\} \cup \{v\}$ or $D_2{}' = \{u_i : 1 \leq i \leq n\} \cup \{v\} = D$ (see Figure 2(a) and 2(b)). Thus, $\gamma_{wfd}(H_n) \geq n + 1$. This proves the equality. ∎

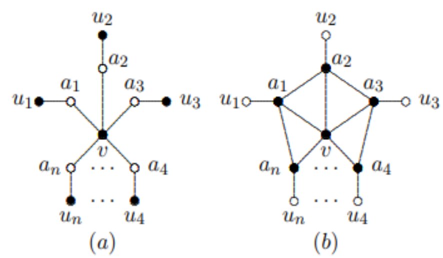

Figure 2: (a) The subgraph $\langle D_1 \rangle_w$ and (b) the subgraph $\langle D_2' \rangle_w$

A nontrivial *complete* graph K_n is the graph in which every two distinct vertices are adjacent. The star graph $K_{1,n}$ is a tree with one internal node and n leaves (but no internal nodes and n + 1 leaves when n ≤ 1). The *fan* $F_{1,n}$ is a graph obtained by joining all the vertices of a path P_n to a further vertex v called the center vertex. The *wheel* $W_{1,n}$ is a graph obtained by joining all vertices of a cycle to a further vertex v called the axial vertex. Figure 3 shows the schematic diagrams of complete graph K_n, star graph $K_{1,n}$, fan graph $F_{1,n}$, and wheel graph $W_{1,n}$.

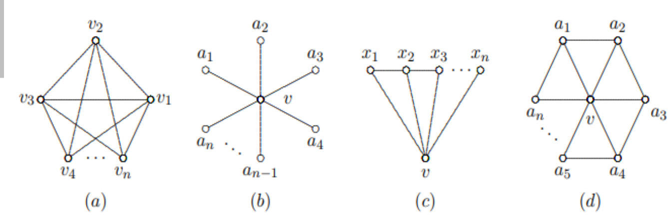

Figure 3: (a) A complete graph K_n; (b) star $K_{1,n}$; (c) fan $F_{1,n}$ and (d) wheel $W_{1,n}$

The next result characterizes the weakly connected fair dominating sets in the join $K_1 + H$ where $K_1 = \langle \{v\} \rangle$ and H is any graph.

Theorem 2.8. Let $K_1 = \langle \{v\} \rangle$ and H be any graph. Then $D \subseteq V(K_1 + H)$ is a wfd-set of $K_1 + H$ if and only if one of the following holds:

(i) $D = \{v\}$

(ii) $D = \{v\} \cup S$, where S is a fair dominating set of H.

(iii) $D \subseteq V(H)$ and D is a $|D|\,fd -$ set of H.

Proof: Suppose that D is a $wfd -$ set of $K_1 + H$. Consider the following cases.

Case 1. Suppose $v \in D$. Then either $D \setminus \{v\} = \phi$ or $D \setminus \{v\} \subseteq V(H)$. If $D \setminus \{v\} = \phi$, then $D = \{v\}$. Suppose $S = D \setminus \{v\} \subseteq V(H)$ and $S = D \setminus \{v\}$ is not a fair dominating set in H. Then D is not a fair dominating set in $K_1 + H$. This contradicts part of the assumption that D is a wfd-set of $K_1 + H$. This proves the necessity of (i) and (ii).

Case 2. Suppose that $v \notin D$. Then $D \subseteq V(H)$. Let $y \in V(H)$. Then $|N_{K_1+H}(y) \cap D| = |D|$. Thus, D is a $|D|\,fd -$ set of $K_1 + H$. Since $D \subseteq V(H)$, we have D is a $|D|\,fd$-set of H.

Conversely, suppose first that $D = \{v\}$. Then D is a dominating set. Let $x \in V(H)$. Then $\left| N_{K_1+H}(x) \cap D \right| = 1$ for every $x \in K_1 + H \setminus D$. The subgraph weakly induced by $D = \{v\}$ is the join $K_1 + H^*$, where H^* is an empty graph with $V(H^*) = V(H)$. Thus, D is a wfd-set in $K_1 + H$. Next, suppose that $D = \{v\} \cup S$, where $S = D \setminus \{v\} \subseteq V(H)$ is a fair dominating set in H. Then D is a dominating set in $K_1 + H$ since $S \subseteq D$. Let $a, b \in D$. Since a and b are either in $V(H)$ or one of a or b coincide with the vertex v, there exists a path joining a and b in $\langle D \rangle_w$ and hence $\langle D \rangle_w$ is connected. Let $w \in V(H) \setminus S$. Then $\left| N_{K_1+H}(w) \cap S \right| = k$ for some integer $k \geq 1$. Thus, $\left| N_{K_1+H}(w) \cap D \right| = \left| N_{K_1+H}(w) \cap (\{v\} \cup S) \right| = k + 1$ for some integer $k \geq 1$. Hence, D is a wfd-set in $K_1 + H$. Lastly, suppose that $D \subseteq V(H)$ is a $|D|fd$-set in H. Then D is a $|D|fd$-set in $K_1 + H$. Let $c, d \in D \subseteq V(H)$. By the adjacency of the vertices in $K_1 + H$, $cv, vd \in E(K_1 + H)$. Thus, $[c, v, d]$ is a $c - d$ path in $\langle D \rangle_w$. Hence, D is a wfd-set in $K_1 + H$. This completes the proof. ∎

The next two corollaries are direct consequences of Theorem 2.8.

Corollary 2.9. Let $K_1 = \langle \{v\} \rangle$ and H be any nontrivial graph. Then $\gamma_{wfd}(K_1 + H) = 1$.

Proof: By Theorem 2.8, $\gamma_{wfd}(K_1 + H)$ is the smallest among the values $|\{v\}|$, where $v \in V(K_1) = \langle \{v\} \rangle$, $1 + \gamma_{fd}(H)$ and k_H where $k_H = \min\{|D| : D \text{ is a } |D|\, fd - \text{set in } H\}$. As a consequence,

$$\gamma_{wfd}(K_1 + H) = \min\{1, 1 + \gamma_{fd}(H), k_H\} = 1. \ \blacksquare$$

Corollary 2.10. For positive integer $n \geq 1$, we have the following weakly connected fair domination numbers of complete graph K_n, star graph $K_{1,n}$, fan graph $F_{1,n}$, and wheel graph $W_{1,n}$.

(i) $\gamma_{wfd}(K_n) = 1, n \geq 1$
(ii) $\gamma_{wfd}(K_{1,n}) = 1, n \geq 2$
(iii) $\gamma_{wfd}(F_{1,n}) = 1, n \geq 2$
(iv) $\gamma_{wfd}(W_{1,n}) = 1, n \geq 3$

A graph G is called *bipartite* if its vertex-set V(G) can be partitioned into two nonempty subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2. The set V_1 and V_2 are called partite sets of G. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called *complete bipartite* graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. The bi-star $(B(r, s))$ for $r, s \geq 2$ is a graph obtained by joining the centers of the stars $K_{1,r}$ and $K_{1,s}$. The *barbell* graph $(B_{n,n})$ is a graph obtained by connecting two complete graphs K_n by a bridge. Figure 4 shows the schematic diagrams of complete bipartite graph $K_{m,n}$, bi-star graph $B(r, s)$, and barbell graph $B_{n,n}$.

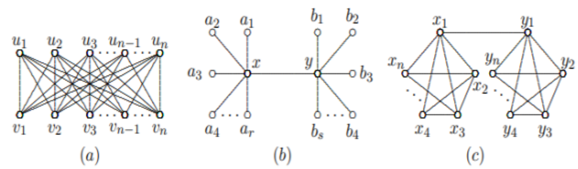

Figure 4: (a) A complete bipartite $K_{m,n}$; (b) bi-star graph $B(r, s)$; and (c) barbell graph $B_{n,n}$

The next result characterizes those graphs with weakly connected fair domination number exactly equal to 2.

Theorem 2.11 Let G be a connected graph of order $n \geq 4$ with $\gamma(G) \neq 1$. Then $\gamma_{wfd}(G) = 2$ if and only if there are distinct vertices $u, v \in V(G)$ such that $(N(u) \setminus \{v\}) \cup (N(v) \setminus \{u\}) = V(G) \setminus \{u, v\}$ and either one of the following holds.

(i) $N(u) \cap N(v) = V(G) \setminus \{u, v\}$
(ii) $(N(u) \setminus \{v\}) \cap (N(v) \setminus \{u\}) = \emptyset$ and $uv \in E(G)$.

Proof: Suppose $\gamma_{wfd}(G) = 2$. Then $\gamma_{wfd}(G) \neq 1$. Thus, $\gamma(G) \neq 1$ by Theorem 2.4. Let $D = \{u, v\}$ be a γ_{wfd}-set in G. Then $(N_G(u) \setminus \{v\}) \cup (N_G(v) \setminus \{u\}) = V(G) \setminus D$. Consider the following cases:

Case 1. D is a $|D|\, fd -$ set of G. That is, D is a $2wfd -$ set of G. Then $|N_G(a) \cap D| = |D| = 2$ for all $a \in V(G) \setminus D$, that is $N_G(a) \cap D = D$ for all $\in V(G) \setminus D$. This means that $N_G(u) \cap N_G(v) = V(G) \setminus D = V(G) \setminus \{u, v\}$.

Case 2. D is a $1fd -$ set of G. That is D is $1\, wfd -$ set of G. Then $|N_G(a) \cap D| = 1$ for all $a \in V(G) \setminus D$. It follows that $N_G(a) \cap D = \{u\}$ or $N_G(a) \cap D = \{v\}$. Thus, $(N_G(u) \setminus \{v\}) \cap (N_G(v) \setminus \{u\}) = \emptyset$. Since $\langle D \rangle_w$ is connected, $uv \in E(G)$.

Conversely, suppose that there exists $u, v \in V(G)$, $u \neq v$ such that $(N_G(u) \setminus \{v\}) \cup (N_G(v) \setminus \{u\}) = V(G) \setminus \{u, v\}$. Suppose first that $N_G(u) \cap N_G(v) = V(G) \setminus \{u, v\}$. Then $D = \{u, v\}$ is a $2\, fd -$ set of G. Also, for every $a \in V(G) \setminus \{u, v\}$ we have $au, av \in E(G)$. Thus, $\langle D \rangle_w$ is connected. Next, suppose that $(N_G(u) \setminus \{v\}) \cap (N_G(v) \setminus \{u\}) = \emptyset$ and $uv \in E(G)$. Then $D = \{u, v\}$ is a $1\, fd -$ set of G. Clearly, $\langle D \rangle_w$ is connected. It follows that D is a $wfd -$ set in G. Hence, $\gamma_{wfd}(G) \leq |D| = 2$. Since $\gamma_{wfd}(G) \neq 1$, it means that $\gamma_{wfd}(G) \geq 2$. Consequently, $\gamma_{wfd}(G) = 2$. ∎

The next five results follow directly Theorem 2.11. The first three are presented without proofs.

Corollary 2.12 Let G be any connected graph of order at least 4 such that $\gamma(H) \neq 1$. If $G = \overline{K_2} + H$, then $\gamma_{wfd}(G) = 2$.

Corollary 2.13 Let G and H be any two graph such that $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. Then $\gamma_{wfd}(G + F) = 2$, where $F = \overline{K_2} + H$.

Corollary 2.14 Let $B_{n,n}$ be a barbell graph with $n \geq 3$. Then $\gamma_{wfd}(B_{n,n}) = 2$.

Corollary 2.15 Let $K_{m,n}$ be a complete bipartite graph with $m, n \geq 2$. Then $\gamma_{wfd}(K_{m,n}) = 2$

Proof: Let A and B be the two partite sets of $K_{m,n}$. Let $D \subseteq V(K_{m,n})$ be such that $|D \cap A| = 1$ and $|D \cap B| = 1$. Then $|D| = 2$. Let $x \in V(K_{m,n}) \setminus D$. Then $x \in A \setminus (D \cap A)$ or $x \in B \setminus (D \cap B)$. Suppose $x \in A \setminus (D \cap A)$. This implies that $|(D \cap B) \cap N_{K_{m,n}}(x)| = 1$. Thus, $|D \cap N_{K_{m,n}}(x)| = 1$, Similarly, if $x \in B \setminus (D \cap B)$ then $|(D \cap A) \cap N_{K_{m,n}}(x)| = 1$. Thus, $|D \cap N_{K_{m,n}}(x)| = 1$. Hence D is 1FD set in $K_{m,n}$. Since $K_{m,n} \cong \overline{K_m} \vee \overline{K_n}$ and $D = (D \cap A) \cup (D \cap B) \subseteq V(\overline{K_m} \vee \overline{K_n})$, by Remark 1.1, $\langle D \rangle_w$ is connected. This means that D is wfd-set in $K_{m,n}$. Therefore, $\gamma_{wfd}(K_{m,n}) \leq |D| = 2$. Next, let D^* be a γ_{wfd}-set in $K_{m,n}$. Suppose $|D^*| = 1$, take $D^* = \{x\}$. Let $x \in D^* \cap A$. Since $m \geq 2$, there exist $y \in A \setminus D$ such that $xy \notin E(K_{m,n})$. This a contradiction to the assumption that D^* is dominating set in $K_{m,n}$. Thus, $\gamma_{wfd}(K_{m,n}) = |D^*| \geq 2$. Therefore, $\gamma_{wfd}(K_{m,n}) = 2$. ■

Corollary 2.16 Let $B(r,s)$ be a bi-star graph with $r,s \geq 2$. Then $\gamma_{wfd}(B(r,s)\ ) = 2$.

Proof: Let x,y be the central vertices of $K_{1,r}$ and $K_{1,s}$, respectively. Then $(N_{B(r,s)}(x) - \{y\}) \cup (N_{B(r,s)}(y) \setminus \{x\} = V(B(r,s)) \setminus \{x,y\}$ with $(N_{B(r,s)}(x) \setminus \{y\}) \cap (N_{B(r,s)}(y) \setminus \{x\} = \emptyset$ and $xy \in E(B(r,s))$. Moreover, it can be observed that $\gamma(B(r,s)) \neq 1$. By Theorem 2.11, $\gamma_{wfd}(B(r,s)\ ) = 2$. ■

Let G and H be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The *vertex corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and n_1 copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. For every vertex $v \in V(G)$, denoted by H^v the copy of H whose vertices are attached one by one to the vertex V. The subgraph of $G \circ H$ corresponding to the join $\langle \{v\} + H \rangle$ is denoted by $v + H$. The *edge corona* of G and H is the graph $G \diamond H$ obtained by taking one copy of G and m_1 copies of H and then joining the two end-vertices of the ith edge of G to every vertex in the ith copy of H. If $ab \in E(G)$, then the copy H whose vertices are connected one by one to both a and b in $G \diamond H$ is called the ab-copy of H and is denoted by H^{ab}. If $V(H) = \{v_1, v_2, \ldots, v_n\}$, then the vertices of H^{ab} may be denoted by ${v_1}^{ab}, {v_2}^{ab}, \ldots, {v_n}^{ab}$.

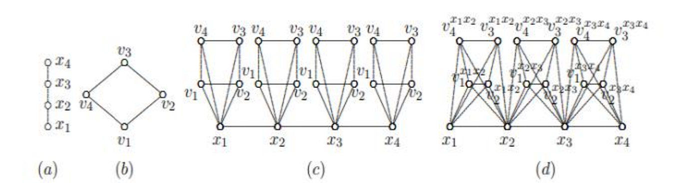

Figure 5: (a) path P_4; (b) cycle C_4; (c) vertex corona $P_4 \circ C_4$ and (d) edge corona $P_4 \diamond C_4$

Theorem 2.17 provides characterization of the wfd-set in the vertex corona of graphs.

Theorem 2.17. Let G be a connected graph and H be any graph. Then $D \subseteq V(G \circ H\ )$ is a wfd-set in $G \circ H$ if and only if one of the following holds:

(i) $D = V(G)$

(ii) $D = V(G) \cup \left(\bigcup_{v \in V(G)} S^v\right)$, where S^v is a $(k-1)fd$-set of H^v for some positive integer $k \geq 2$.

Proof: Let D be a $kwfd$-set of $G \circ H$. Let $D = X \cup Y$, where $X \subseteq V(G)$ and $Y \subseteq \bigcup_{v \in V(G)} V(H^v)$. We consider the following cases:

Case 1. $Y = \emptyset$. Then $D = X$. Since D is a weakly connected dominating set and fair dominating set in $G \circ H$, it follows that $D = V(G)$.

Case 2. $Y \neq \emptyset$. Clearly, $X \neq \emptyset$ since D must be a weakly connected dominating set in $G \circ H$. So, $X = D \cap V(G)$. Let $v \in X$. Suppose there exists $u \in N_G(v) \setminus X$ and let $D_u = V(H^u) \cap D$. Since $u \notin X$ and D is a $kwfd$-set, it follows that D_u is a kfd-set in H^u. This means that $|N_{G \circ H}(u) \cap D| = |N_G(u) \cap X| + |D_u| \geq 1 + |D_u| > k$. This is contrary to the assumption that D is a kfd-set. Thus, $u \in X$ for every $u \in N_G(v)$. Thus, $V(G) \subseteq D$ since G is connected. Hence, $X = V(G)$ and so $D = V(G) \cup Y$. If $D_v = V(H^v)$ for all $v \in V(G)$, take $k \geq 2$. On the other hand, suppose $D_v \neq V(H^v)$ for some v, then take $k \geq 2$. Let $x \in Y$ such that $x \in V(H^v)$ for some $v \in V(G)$. Let $D_v = V(H^v) \cap D$. Then $x \in D_v$. Since $v \in D$ and D is a $kwfd$-set, it follows that D_v is a $(k-1)fd$-set in H^v. Therefore, $Y = \bigcup_{v \in V(G)} D_v$, where each D_v is a $(k-1)fd$-set in H^v.

Conversely, suppose first that $D = V(G)$. Then for $a, b \in \bigcup_{v \in V(G)} V(H^v)$, we have $|N_{G \circ H}(a) \cap D| = |\{v\}| = 1 = |N_{G \circ H}(b) \cap D|$. Thus, D is a $1fd$-set of $G \circ H$. Now, if (b) holds, then for any $c, d \in \bigcup_{v \in V(G)} V(H^v) \setminus D$, $|N_{G \circ H}(c) \cap D| = (k-1) + 1 = k = |N_{G \circ H}(d) \cap D|$. It follows that D is a kfd-set of $G \circ H$. Since $V(G) \subseteq D$, in both cases, $\langle D \rangle_w$ is connected. Therefore, D is a $kwfd$-set in $G \circ H$. ■

Corollary 2.18. Let G be a connected graph and let H be any graph. Then $\gamma_{wfd}(G \circ H\ ) = |V(G)|$.

Proof: By Theorem 2.17, $\gamma_{wfd}(G \circ H\ )$ is the minimum among the values $|V(G)|$ and $|V(G)| + M$, where $M = \sum_{v \in V(G)} |S^v|$ and S^v is a $(k-1)fd$-set of H^v for some integer $k > 1$. As a consequence, $\gamma_{wfd}(G \circ H\ ) = |V(G)|$. ■

We now examine the edge corona of two graphs G and H.

Theorem 2.19. Let G be any connected graph of order $n \geq 2$, size $m \geq 1$, and H be any graph. Then $\gamma_{wfd}(G \diamond H\ ) = 1$ if and only if every edge in G is incident to a commom vertex in G.

Proof: Suppose first that $\gamma_{wfd}(G \diamond H\ ) = 1$. Let $D = \{v\}$ be a γ_{wfd}-set in $G \diamond H$. Suppose that there exists an edge, say ab $\in E(G)$ with ab not incident to vertex $v \in V(G)$. Since $|V(H)| \geq$

1, there exists a vertex $x \in V(H^{ab})$ such that $vx \notin E(G \diamond H)$. It means that x is not dominated by D in $G \diamond H$. This contradicts to the assumption that D is dominating set in $G \diamond H$.

Conversely, suppose that every edge in G is incident to a vertex $v^* \in V(G)$. Let $D^* = \{v^*\}$. Then D^* is a dominating set in G. Since v^* is incident to every edge in G, it follows that every vertex $x^* \in V(H^{v^*z})$ is also adjacent to v^* with $z \in V(G)$. Thus, D^* is a dominating set in $G \diamond H$. Next, since $|D^*| = 1$, every edge in G is incident to $v^* \in D^*$, clearly $\langle D^* \rangle_w$ is connected. In fact, $\langle D^* \rangle_w \cong K_{1,r}$, where $r = n - 1 + m|V(H)|$. Now, let $u, w \in V(G \diamond H) \setminus D^*$. Then $u \neq v^*$ and $w \neq v^*$. Thus, $|N_{G \diamond H}(u) \cap D^*| = 1 = 1|N_{G \diamond H}(w) \cap D^*|$. Hence, D^* is a $1fd$-set in $G \diamond H$. Therefore, D^* is a wfd-set in $G \diamond H$. As a consequence, $\gamma_{wfd}(G \diamond H) \leq |D^*| = 1$. By Remark 2.1, $\gamma_{wfd}(G \diamond H) \geq 1$. Consequently, $\gamma_{wfd}(G \diamond H) = 1$. This completes the proof. ■

The following corollaries are direct consequences of Theorem 2.19.

Corollary 2.20 If G is a nontrivial connected graph and H is any graph with $\gamma_{wfd}(G \diamond H) = 1$, then $\gamma(G) = 1$.

Corollary 2.21 Let H be any graph. If G is a star graph, then $\gamma_{wfd}(G \diamond H) = 1$.

The next result provides sufficient conditions for the wfd-set in the edge corona $G \diamond H$

Theorem 2.22 Let G be any connected graph of order $n \geq 4$ and H be any graph. Then $D \subseteq V(G \diamond H)$ is a wfd-set in $G \diamond H$ if one of the following holds:

(i) $D = V(G)$

(ii) $D = V(G) \cup S$, where $S = \bigcup_{\substack{a,b \in V(G) \\ ab \in E(G)}} S^{ab}$ and S^{ab} is a $(k-2)fd$-set in H^{ab} for some $k \geq 3$, $k \in \mathbb{N}$.

Proof: Suppose (i) holds. Then for $x, y \in \bigcup_{ab \in E(G)} V(H^{ab})$, $|N_{G \diamond H}(x) \cap D| = |\{a, b\}| = 2 = |N_{G \diamond H}(y) \cap D|$. Thus, D is a $2wfd$-set in $G \diamond H$. Next, suppose (ii) holds. Then

$$\begin{aligned} |N_{G \diamond H}(x') \cap D| &= (k-2) + |\{a, b\}| \\ &= (k-2) + 2 \\ &= |N_{G \diamond H}(y') \cap D| \end{aligned}$$

for any $x'y' \in V(H^{ab}) \setminus D$. Thus, D is a kfd-set in $G \diamond H$. Since $V(G) \subseteq D$ and G is connected, $\langle D \rangle_w$ is connected in both cases. Therefore, D is a $kwfd$-set in $G \diamond H$. ■

Corollary 2.23 Let G be any graph of order $n \geq 4$ and H be any graph. Then $\gamma_{wfd}(G \diamond H) \leq n$.

Proof: By Theorem 2.22, $\gamma_{wfd}(G \diamond H)$ is at most the minimum between $|V(G)|$ and $|V(G)| + |E(G)||S^{ab}|$, where S^{ab} is a $(k-2)fd$-set in H^{ab} for every $ab \in E(G)$ and some $k \geq 3, k \in \mathbb{N}$. That is, $\gamma_{wfd}(G \diamond H) \leq \min\{|V(G)|, |V(G)| + |E(G)||S^{ab}|\} = |V(G)|$. ■

CONCLUSION AND RECOMMENDATION

For any connected graph $G = (V(G), E(G))$, a dominating set $D \subseteq V(G)$ is a weakly connected k-fair dominating set in G, abbreviated $k$wfd – set, if the subgraph $\langle D \rangle_w = (N_G[D], E_w)$ is connected, where E_w is the set of all edges in G with at least one vertex in D and $|N_G(u) \cap D| = k$ for every $u \in V(G) \setminus D$ for some integer $k \geq 1$. This article focused on the weakly connected k-fair domination of some special graphs, join, vertex and edge coronas of graphs. For future research, it would be interesting to investigate the parameter for some other binary operations on graphs such as cartesian products, lexicographic products, and obtained Nordhaus-Gaddum-type results.

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